Exercise 7
Canonical forms, pole placement and uncertain systems

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INTRODUCTION
Canonical forms and pole placement were previously discussed in the lecture. The exercise addresses their application and implementation. Additionally, stability analysis for uncertain systems are discussed.

PROBLEMS
1. Consider the LTI system
\[ \dot{x} = Ax + bu \]
with \( n \) states but only one input. Assume that the system is controllable. Then there exists a state transformation matrix \( T \) such that the transformed system
\[ \dot{z} = \hat{A}z + \hat{b}u \]
with the states \( z = Tx \) is in control canonical form, i.e.,
\[
\hat{A} = \begin{pmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 1 \\
-a_1 & -a_2 & \ldots & -a_n \\
\end{pmatrix}
\]
and
\[
\hat{b} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 \\
\end{pmatrix},
\]
where \( a_1, \ldots, a_n \) are the coefficients of the characteristic polynomial of \( A \).
Show that the transformation \( T \) can be chosen as:
\[
T = \begin{pmatrix}
\varepsilon_n^{T} C^{-1} \\
\vdots \\
\varepsilon_n^{T} C^{-1} A^{n-1} \\
\end{pmatrix},
\]
where
\[
\mathcal{C} = \begin{pmatrix} b & Ab & \cdots & A^{n-1}b \end{pmatrix} \in \mathbb{R}^{n \times n},
\]
\[
e_n^T = (0 \cdots 0 1).
\]

**Hint:** Note that the Cayley-Hamilton theorem states that
\[
A^n + a_{n-1}A^{n-1} + \cdots + a_2A + a_1I = 0.
\]

**Extra Credit.** Derive the state-space transformation \( \tilde{T} \) so that \( \tilde{z} = \tilde{T}x \) and
\[
\tilde{A} = \begin{pmatrix} 0 & 0 & \cdots & -a_n \\ 1 & 0 & \cdots & -a_2 \\ 0 & 1 & \cdots & -a_3 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ & & & -a_n \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

Use the fact that \( \mathcal{C}^{-1}\mathcal{C} = I \)

2. Consider the LTI system
\[
\dot{x} = Ax + Bu
\]
with \( n \) states and \( m \) inputs. Propose a method for designing a state feedback controller \( u = -Kx \) such that all of the eigenvalues \( \lambda_i \) of the closed-loop system are stable, i.e.,
\[
\text{real}(\lambda_i) < 0 \quad \forall i \in \{1, \ldots, n\},
\]
and the imaginary parts of all closed-loop eigenvalues are smaller than \( \beta > 0 \), i.e.,
\[
|\text{imag}(\lambda_i)| < \beta \quad \forall i \in \{1, \ldots, n\}.
\]

**Hints:** Try to describe the set
\[
\mathcal{D} = \{z \in \mathbb{C} | \text{real}(z) < 0, |\text{imag}(z)| < \beta\}
\]
as \( \mathcal{D} = \{z \in \mathbb{C} | f_{\mathcal{D}}(z) < 0\}, \) where
\[
f_{\mathcal{D}}(z) = L + Mz + M^Tz^*
\]
with \( L, M \in \mathbb{R}^{3 \times 3} \). As a preparation, first consider the set
\[
\hat{\mathcal{D}} = \{z \in \mathbb{C} | |\text{imag}(z)| < \beta\}.
\]
Then, include the result for \( \text{real}(z) < 0 \) known from the lecture.
3. Consider the time-variant autonomous system

\[ \dot{x} = A(t) x, \]

where \( A(t) \in \Omega \subset \mathbb{R}^{n \times n} \) for all \( t \geq 0 \). The set \( \Omega \) can be viewed as an uncertainty set which, for example, accounts for imprecisely known system dynamics.

Suppose further that the set \( \Omega \) is the convex hull of a collection of matrices \( A_1, \ldots, A_\nu \) with \( \nu \in \mathbb{N} \), i.e.,

\[ \Omega = \text{conv}\{A_1, \ldots, A_\nu\}. \]

Show that the system above is stable if there exists some \( P \) such that \( P \succ 0 \) and

\[ A_i P + P A_i^T \prec 0 \quad \forall i \in \{1, \ldots, \nu\}. \]

Explain why this is different than requiring that each of the matrices \( A_i \) is stable.

**Hints:** Search for stable matrices \( A_1 \) and \( A_2 \) which are such that \( 0.5A_1 + 0.5A_2 \) is unstable.