Exercise 2
Linearization and Case Study (I)

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INTRODUCTION

Usually, models describing real-life systems are nonlinear. However, linear models can often be used to locally approximate the system behavior with sufficient accuracy. The exercise deals with the linearization of nonlinear systems and the analysis of the linear approximation.

PROBLEMS

1. We begin by verifying the property of superposition for linear state space systems of the form

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t).
\end{align*}
\]

Depending on the input \(u(t)\) and the initial state \(x(0) = x_0\), the output \(y(t)\) of system (1)–(2) can be described as

\[
y(t) = Ce^{At}x_0 + C \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t).
\]

Now, consider two input-output pairs \((u_1(t), y_1(t))\) and \((u_2(t), y_2(t))\) and show that the output to the superposed input

\[
u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)
\]

with \(\alpha_1 + \alpha_2 = 1\) is given by \(y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)\).
2. Linearize the model

\[
\begin{bmatrix}
(M + m) & -ml \cos \theta \\
-ml \cos \theta & ml^2
\end{bmatrix}
\begin{bmatrix}
\ddot{p} \\
\ddot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
ml \sin \theta \dot{\theta}^2 \\
-mgl \sin \theta
\end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F
\]

for the inverted pendulum on a moving cart from Exercise 1 around the equilibrium \( p_0 = 0, \theta_0 = 0, \dot{p}_0 = 0, \) and \( \dot{\theta}_0 = 0. \) Then write the linearized system as a state-space system of the form (??)-(??) with \( u = F, \; x = (q, \dot{q})^T, \) and \( y = q, \) where \( q = (p, \theta)^T. \)

Hints: Remember the fact that in a neighbourhood of \( \theta = 0 \) we may substitute \( \sin \theta \sim \theta, \cos \theta \sim 1, \dot{\theta}^2 = o(\dot{\theta}). \)

Hints: Write the dynamics in the form \( \dot{x} = f(x) + g(x)u \) with \( f : \mathbb{R}^4 \to \mathbb{R}^4 \) and \( g : \mathbb{R}^4 \to \mathbb{R}^4. \) Then compute the Jacobians \( A = \frac{\partial f}{\partial x} \) and \( B = \frac{\partial g}{\partial x}. \) Note that the structure of \( A \) and \( B \) is as follows:

\[
\dot{x} = \begin{bmatrix}
0 & 0 & * & 0 \\
0 & 0 & * & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
* \\
*
\end{bmatrix} F.
\]

3. Linearize the model

\[
\begin{align*}
\dot{h}(t) &= bh(t) - ah(t) \ell(t) \\
\dot{\ell}(t) &= ch(t) \ell(t) - d \ell(t)
\end{align*}
\]

for the predator-prey system from Exercise 1 around the states \( x^{(3)}_0 = (5, 5)^T \) and \( x^{(4)}_0 = (2, 1)^T \) and compute the eigenvalues of the linearized system.

4. Case Study (I): We study the longitudinal dynamics of an aircraft (see Fig. ??).

Figure 1: Body fixed coordinates (left, see [?], Fig. 1.10) and longitudinal dynamics (right, see [?], Fig. 3.4] of an aircraft.
According to [? p. 108], the behavior of the aircraft can be described by the following (coupled differential) equations:

\[
\begin{align*}
\Delta \dot{u} - X_u \Delta u - X_w \Delta w + g \cos \theta_0 \Delta \dot{\theta} &= X_\delta \Delta \delta_e + X_{\delta_T} \Delta \delta_T, \\
-Z_u \Delta u + \Delta \dot{w} - Z_w \Delta w - u_0 \Delta \dot{\theta} - g \sin \theta_0 \Delta \dot{\theta} &= Z_\delta \Delta \delta_e + Z_{\delta_T} \Delta \delta_T, \\
-M_u \Delta u - M_w \Delta \dot{w} + M_w \Delta w + \Delta \dot{\theta} - M_q \Delta \dot{\theta} &= M_\delta \Delta \delta_e + M_{\delta_T} \Delta \delta_T,
\end{align*}
\]

where the inputs \( \Delta \delta_T \) and \( \Delta \delta_e \) denote the trust and the elevator angle, respectively. The variables \( \Delta u \) and \( \Delta w \) are components of the velocity along \( x \) and \( z \) axis, respectively, and \( \Delta \dot{\theta} \) is the pitch angle. The so-called force derivatives \( X_u, X_w, X_\delta, X_{\delta_T}, Z_u, Z_w, Z_\delta, Z_{\delta_T}, M_u, M_w, M_q, M_w, M_{\delta_T}, M_{\delta_T} \) are defined by the aerodynamic forces which depend on the aircraft properties and the operating conditions as

\[
\begin{align*}
X_u &= -(C_{D_u} + 2C_{D_s})QS \frac{u_0}{m}, & Z_u &= -(C_{L_u} + 2C_{L_s})QS \frac{u_0}{m}, & M_u &= 0, \\
X_q &= 0, & Z_q &= 0, & M_q &= \frac{C_{m_q} QS \bar{e}^2}{2u_0 I_y}, \\
X_w &= -(C_{D_w} + 2C_{D_s})QS \frac{u_0}{m}, & Z_w &= -(C_{L_w} + 2C_{L_s})QS \frac{u_0}{m}, & M_w &= \frac{C_{m_w} QS \bar{e}}{u_0 I_y}, \\
X_\delta &= 0, & Z_\delta &= -\frac{C_{Z_\delta} QS}{m}, & M_\delta &= \frac{C_{m_\delta} QS \bar{e}}{I_y}, \\
X_{\delta_T} &= 0, & Z_{\delta_T} &= 0, & M_{\delta_T} &= \frac{C_{m_{\delta_T}} QS \bar{e}^2}{2u_0^2 I_y},
\end{align*}
\]

with dynamic pressure \( Q = 0.5 \rho u_0^2 \), air density \( \rho \), average chord \( \bar{e} \), wing area \( S \), mass \( m \), moment of inertia \( I_y \), and flight velocity \( u_0 \). Have a look at the provided excerpt from [?] for details on the aircraft model.

a) Rewrite the equations of motions (??)–(??) in terms of a linear state-space system \( \dot{x} = Ax + Bu \).

**Hints:** Introduce \( q = \dot{\theta} \) and consider the state vector \( x = (\Delta u, \Delta w, \Delta q, \Delta \theta)^T \).

For a Boeing 747, in level flight, 40000ft, 774 ft/sec, we obtain the linear system

\[
\dot{x} = \begin{pmatrix}
-0.003 & 0.039 & 0 & -0.322 \\
-0.065 & -0.319 & 7.747 & 0 \\
0.020 & -0.101 & -0.429 & 0 \\
0 & 0 & 1.000 & 0
\end{pmatrix} x + \begin{pmatrix}
0.01 & 1.00 \\
-0.18 & -0.04 \\
-1.16 & 0.60 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\Delta \delta_e \\
\Delta \delta_T
\end{pmatrix}
\]

according to [? p. 151] (see also [? p. 149] for the linear parametric model).

b) With MATLAB, compute the eigenvalues of the state-space representation matrix \( A \). Plot the eigenvalues in the complex plane.
c) With $\Delta \delta_e = 0$ and $\Delta \delta_T = 0$ simulate the response for the initial condition
$\Delta u(0) = 0, \Delta w(0) = 0, \Delta q(0) = 0, \Delta \theta(0) = 0.1\text{rad}$.

d) Verify, via simulation, the effect of a constant disturbance on the vertical component of the velocity of magnitude $d = 3.5 \text{ft/s}$ for 5 s (i.e., for $t \in [0, 5]$).

Hints: Consider the disturbed system $\dot{x} = A \begin{pmatrix} x_1 & x_2 + d & x_3 & x_4 \end{pmatrix}^T$ and try to rewrite this in the form $\dot{x} = Ax + Ed$.

e) Interpret the response of the system in terms of its eigenvalues. Which mode is dominant? Compute the damping factor of each mode.

Hints: First use the MATLAB function \texttt{damp}. Then try to reproduce the computed damping factors by analyzing the eigenvalues of $A$.

REFERENCES
