CDT
Autonomous and Intelligent Machines and Systems

Introduction to Modern Control

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Control for Linear Time-Invariant Systems

Given a system in the form

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

- How do we determine what the state \( x \) is, given only the output \( y \)?
- How can we design a stabilising controller?
- How can we place the (stabilized) eigenvalues at specified locations?
- How do we place the (stabilized) eigenvalues in specified regions?
- Which controller is the ‘best’ according to some design objective?

Will address all of these questions using Linear Matrix Inequalities (LMIs) and Semidefinite Programs (SDPs).
• The closed-loop state dynamics are

\[ \dot{x} = (A - BK)x \]

• The system is \textit{stable} if the eigenvalues of \((A-BK)\) all have negative real parts.

• The state \(x\) is usually not directly observable, i.e. usually \(C \neq I\).
Finding a State Observer

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

\[ \dot{\hat{x}} = A\hat{x} + Bu + Le \]
\[ \hat{y} = C\hat{x} + Du \]

- Use an observer to reconstruct the state \( x \) from the plant outputs \( y \).
- The closed-loop dynamics are:

\[
\begin{pmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{pmatrix} =
\begin{bmatrix}
A & 0 \\
LC & (A - LC')
\end{bmatrix}
\begin{pmatrix}
x \\
\hat{x}
\end{pmatrix}
+ \begin{bmatrix}
B \\
B
\end{bmatrix} u
\]

- The observer is stable if the eigenvalues of \( (A - LC) \) all have negative real parts.

\[
\tilde{x} := x - \hat{x} \quad \implies \quad \dot{\tilde{x}} = (A - LC')\tilde{x}
\]
• Use the \textit{estimated} state in place of the \textit{true} state for the controller.

• The closed-loop dynamics are:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{bmatrix} = \begin{bmatrix}
A & -BK \\
LC & A - LC - BK
\end{bmatrix}\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix} \iff \begin{bmatrix}
\dot{x} \\
\dot{\tilde{x}}
\end{bmatrix} = \begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix}\begin{bmatrix}
\tilde{x}
\end{bmatrix}
\]
Joint Controller / Observer Design (block form)

Plant

Controller

Observer
**Fact**: Any controllable SISO system has state-space representation \((A,B,C,D)\) in the form:

\[
A = \begin{bmatrix}
-a_1 & -a_2 & -a_3 & \cdots & -a_n \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix} 1 \\
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}, \quad (C, D) \text{ unstructured}
\]

The poles of this system are the roots of the polynomial equation

\[
\det(sI - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_{n-1} s + a_n = 0
\]

**Claim**: It is possible to choose control gain \(K\) such that the loop poles of \((A-BK)\) are placed at arbitrarily chosen positions.
**Fact**: Any observable SISO system has state-space representation \((A,B,C,D)\) in the form:

\[
A = \begin{bmatrix}
-a_1 & 1 & 0 & \cdots & 0 \\
-a_2 & 0 & 1 & \cdots & 0 \\
-a_3 & 0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
-a_n & 0 & \cdots & 0 & 0 \\
\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}
\]

\((B, D)\) unstructured

The poles of this system are the roots of the polynomial equation

\[
\det(sI - A) = s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n = 0
\]

**Claim**: It is possible to choose control gain \(L\) such that the loop poles of \((A-LC)\) are placed at arbitrarily chosen positions.
Lyapunov’s Stability Theorem for Linear Systems

The linear system \( \dot{x} = Ax \) is exponentially stable iff there exists some matrix \( P > 0 \) such that

\[
AP + PA^T < 0.
\]

Equivalently, it is exponentially stable iff there exists some matrix \( Q > 0 \) such that

\[
QA + A^T Q < 0.
\]

**For Controller Design:**

\[
(A - BK)P + P(A - BK)^T < 0
\]

\[
\downarrow
\]

Change of variables: \( F := -KP \)

\[
\downarrow
\]

\[
AP + BF + PA^T + F^T B^T < 0
\]

**For Observer Design:**

\[
Q(A - LC) + (A - LC)^T Q < 0
\]

\[
\downarrow
\]

Change of variables: \( G := -QL \)

\[
\downarrow
\]

\[
QA + GC + A^T Q + B^T G < 0
\]
Controller Design using LMIs

- Recall the joint dynamics for an observer and controller pair:

\[
\begin{pmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{pmatrix} =
\begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix}
\begin{pmatrix}
x \\
\hat{x}
\end{pmatrix}
\]

**Claim:**

1) There exists a pair \((L,K)\) that stabilizes the above system iff there exists \((Q,P)\) satisfying the following LMI conditions:

\[
AP + BF + PA^T + F^T B^T < 0
\]

\[
QA + GC + A^T Q + B^T G < 0
\]

\[P > 0, Q > 0\]

2) The eigenvalues of the above system are \(\text{eig}(A - LC) \cup \text{eig}(A - BK)\).
The Kronecker product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is defined as

$$A \otimes B = \begin{bmatrix}
A_{11}B & A_{12}B & \cdots & A_{1m}B \\
A_{21}B & A_{22}B & & \\
& \ddots & \ddots & \\
A_{n1}B & \cdots & \cdots & A_{nm}B
\end{bmatrix}$$

where $A \otimes B \in \mathbb{R}^{mp \times nq}$.

Some useful Kronecker product identities:

$$(A + B) \otimes C = A \otimes C + B \otimes C$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$(A \otimes B)^T = A^T \otimes B^T$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$\exists S$ such that: $$(A \otimes B) = S^T (B \otimes A) S$$
Defining Pole Placement Regions

Choose matrices $L = L^T \in \mathbb{R}^{p \times p}$ and $M \in \mathbb{R}^{p \times p}$ define a function $f_D : \mathbb{C} \rightarrow S^{p \times p}$ as

$$f_D(z) := L + zM + z^*M^T$$

Use this to define a region of the complex plane:

$$\mathcal{D} := \{ z \in \mathbb{C} \mid f_D(z) < 0 \}$$

Example: real$(z) < -\alpha$ [open half plane]

$$f_D(z) = 2\alpha + z + z^* \quad (L = 2\alpha, \quad M = 1)$$
Defining Pole Placement Regions

Choose matrices \( L = L^T \in \mathbb{R}^{p \times p} \) and \( M \in \mathbb{R}^{p \times p} \) define a function \( f_D : \mathbb{C} \rightarrow \mathbb{S}^{p \times p} \) as

\[
f_D(z) := L + zM + z^* M^T
\]

Use this to define a region of the complex plane:

\[
D := \{ z \in \mathbb{C} \mid f_D(z) \prec 0 \}
\]

Example: \( |z + q| < r \) \quad [disk of radius \( r \) centered at \((-q, 0)]

\[
f_D(z) = \begin{bmatrix} -r & q + z \\ q + z^* & -r \end{bmatrix}, \quad \text{so} \quad L = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]
Defining Pole Placement Regions

Choose matrices $L = L^T \in \mathbb{R}^{p \times p}$ and $M \in \mathbb{R}^{p \times p}$
define a function $f_D : \mathbb{C} \rightarrow S^{p \times p}$ as

$$f_D(z) := L + zM + z^* M^T$$

Use this to define a region of the complex plane:

$$\mathcal{D} := \{ z \in \mathbb{C} \mid f_D(z) < 0 \}$$

Example: $z$ within a conic sector with inner angle $2\theta$

$$f_D(z) = \begin{bmatrix} (z + z^*) \sin \theta & (z - z^*) \cos \theta \\
(z^* - z) \cos \theta & (z + z^*) \sin \theta \end{bmatrix}$$
Given $A \in \mathbb{R}^{n \times n}$ and $P = P^T \in \mathbb{R}^{n \times n}$, define

$$M_D(A, P) := L \otimes P + M \otimes (AP) + M^T \otimes (PA^T)$$

**Theorem:** $\text{eig}(A) \subseteq \mathcal{D}$ if and only if there exists $P = P^T$ such that

$$P \succ 0, \quad M_D(A, P) \prec 0$$

**Example:** All closed-loop poles have real part less than $-\alpha$

$$f_D(z) = 2\alpha + z + z^* \quad (L = 2\alpha, \quad M = 1)$$

$$\text{eig}(A_{clp}) \in \mathcal{D} \iff \exists P \succ 0, \quad 2\alpha P + A_{clp}P + PA_{clp}^T \prec 0$$
Unstable system:

\[ \dot{x} = Ax + Bu \]

\[
A = \begin{bmatrix}
1 & 0 & 2 \\
2 & 0 & -4 \\
3 & 5 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

Eigenvalues:

\begin{align*}
+ & 2.4735 \\
- & 0.2368 + 4.0144i \\
- & 0.2368 - 4.0144i
\end{align*}
**Unstable system:**

\[ \dot{x} = Ax + Bu \]

\[ A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & -4 \\ 3 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

**Stability condition:**

\[ AP + BF + PA^\top + F^\top B^\top < 0, \quad P > 0 \]

\[ \downarrow \]

\[ K = -FP^{-1} \]

\[ = \begin{bmatrix} 3.4469 & 4.9015 & 6.2468 \end{bmatrix} \]

**Eigenvalues:**

\[ -0.5755 \]

\[ -0.4357 + 6.2774i \]

\[ -0.4357 - 6.2774i \]
Numerical Example

**Unstable system:**
\[ \dot{x} = Ax + Bu \]

\[ A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & -4 \\ 3 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

**Stability condition:**
\[ AP + BF + PA^T + F^T B^T < 0, \quad P > 0 \]

**All eigenvalues with Real(\(\lambda\)) < -2**

\[ AP + BF + PA^T + F^T B < -4P \]

\[ \Downarrow \]

\[ K = -FP^{-1} \]

\[ = \begin{bmatrix} 8.2445 & 10.1017 & 14.5795 \end{bmatrix} \]

Eigenvalues:
- 2.0840
- 2.0803 + 7.5960i
- 2.0803 - 7.5960i
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Linear Quadratic Performance Objectives

So far everything has related to finding controllers that meet certain design requirements. Which controller is the ‘best’?

**Continuous-time Linear Quadratic Regulator (LQR) Problems**

\[
\min_{u(\cdot)} \int_0^T \left[ x(\tau)^\top Q x(\tau) + u(\tau)^\top R u(\tau) \right] \, d\tau + x(T)^\top Q_f x(T)
\]

subject to: \[ \begin{aligned}
\dot{x} &= Ax + Bu \\

x(0) &= x_0
\end{aligned} \]

- \(Q\) and \(R\) are symmetric positive definite matrices
- \(Q_f\) is the (positive semidefinite) terminal cost
- \(T\) is the horizon length
- The decision variable \(u\) is an input trajectory \(u : [0, T] \to \mathbb{R}^m\)

This problem is an *infinite dimensional* optimisation problem.
Dynamic Programming Solution Method

Define a ‘cost-to-go’ starting at some time $t$:

**Cost to go:**

$$V_t(z) = \min_{u(\cdot)} \int_t^T \left[ x(\tau)^\top Q x(\tau) + u(\tau)^\top Ru(\tau) \right] d\tau + x(T)^\top Q_f x(T)$$

subject to:

$$\dot{x} = Ax + Bu$$

$$x(t) = z$$

- The decision variable $u$ is the partial input trajectory $u : [t, T] \rightarrow \mathbb{R}^m$

- The value function $V_t : \mathbb{R}^n \rightarrow \mathbb{R}$ describes the tail cost, for any time $0 \leq t \leq T$

- At $t = T$, $V_T(z) = z^\top Q_f z$.

**Fact**: The tail cost function $V_t(z) = z^\top P_t z$ is a quadratic function for every $t$. 
Overview of solution method:

- Work backward from time \( t + \delta t \) to time \( t \).
- Assume \( x(t) = z \) at time \( t \).
- Assume constant input \( u(t) = \mu \) over interval \([t, t + \delta t]\).

Total cost will have two components:

- Cost incurred over interval \([t, t + \delta t]\)

\[
\int_t^{t+\delta t} [x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)]d\tau \approx [z^T Q z + \mu^T R \mu] \delta t 
\]

- Cost-to-go from whatever state we land in at time \( t + \delta t \).

\[
V_{t+\delta t} (x(t + \delta t))
\]
Approximate state at time $t + \delta t$:

$$x(t + \delta t) \approx x(t) + \dot{x}\delta t$$

$$= z + (Az + B\mu)\delta t$$

Approximate cost at time $t$:

$$V_t(z) \approx \min\left\{ [z^T Q z + \mu^T R \mu] \delta t + V_{t+\delta t}(x(t + \delta t)) \right\}$$

Approximate cost at time $t + \delta t$:

$$V_{t+\delta t}(x(t + \delta t))$$

$$\approx V_{t+\delta t}(z + (Az + B\mu)\delta t)$$

$$\approx (z + (Az + B\mu)\delta t)^T P_{t+\delta t}(z + (Az + B\mu)\delta t)$$

$$\approx (z + (Az + B\mu)\delta t)^T[P_t + \dot{P}_t \delta t](z + (Az + B\mu)\delta t)$$

$$\approx z^T P_t z + \left[2(Az + B\mu)^T P_t z + z^T \dot{P}_t z\right] \delta t$$
1) Find the minimising $\mu$ over the interval $t + \delta t$:

$$V_t(z) \approx \min_{\mu} \left\{ [z^T Q z + \mu^T R \mu] \delta t + z^T P_t z + \left[ 2(Az + B\mu)^T P_t z + z^T \dot{P}_t z \right] \delta t \right\}$$

$$\downarrow$$

$$\mu^* = -R^{-1} B^T P_t z$$

$\implies$ Optimal controller is

$$u(t) = -R^{-1} B^T P_t x(t) = -K_t x(t)$$

Optimal controller is a (time-varying) linear state feedback controller.
2) Find the time varying value function $P_t$:

$$V_t(z) = z^T P_t z \approx \left\{ \left[ z^T Q z + (\mu^*)^T R \mu^* \right] \delta t + z^T P_t z + \left[ 2(Az + B \mu^*)^T P_t z + z^T \dot{P}_t z \right] \right\} \delta t$$

Plug in $\mu^* = -R^{-1} B^T P_t z$:

$$\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q$$

Optimal cost can be solved by integrating backwards in time, with terminal condition $P_T = Q_f$.

This is a *differential Riccati equation*. Solvable via numerical integration.
Continuous-time Linear Quadratic Regulator (LQR) Problems

\[ V_\infty(x_0) = \min_{u(\cdot)} \int_0^\infty \left[ x(\tau)^\top Q x(\tau) + u(\tau)^\top R u(\tau) \right] d\tau \]

subject to: \[ \dot{x} = Ax + Bu \]
\[ x(0) = x_0 \]

Limiting case as \( T \to \infty \) for the LQR problem.

If we assume that the pair \((A, B)\) is controllable, then \( V \) is finite.

Cost goes to \( V_\infty(z) = z^\top P z \).
Infinite-Horizon LQR Regulator

Continuous-time Linear Quadratic Regulator (LQR) Problems

\[ V_\infty(x_0) = \min_{u(\cdot)} \int_0^\infty \left[ x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right] d\tau \]

subject to : \[ \dot{x} = Ax + Bu \]
\[ x(0) = x_0 \]

- Optimal infinite-horizon cost will be

\[ V(x_0) = x_0^T P x_0 \]

where \( P = \lim_{t \to -\infty} P_t \) is the solution to

\[ A^T P + PA - PBR^{-1}B^T P + Q = 0. \]

- The optimal controller is

\[ u(t) = -K x(t), \quad \text{where } K := R^{-1}B^T P \]
Continuous-time Algebraic Riccati Equation (CARE)

- Must solve a nonlinear matrix equation to get our controller:

\[ A^\top P_{\text{are}} + P_{\text{are}} A - P_{\text{are}} B R^{-1} B^\top P_{\text{are}} + Q = 0 \]

This is the continuous-time Algebraic Riccati Equation (ARE).

- Can also left/right multiply by \( S_{\text{are}} := P_{\text{are}}^{-1} \), to get the alternative form:

\[ S_{\text{are}} A^\top + A S_{\text{are}} - B R^{-1} B^\top + S_{\text{are}} Q S_{\text{are}} = 0 \]

- Or write it together with its associated controller:

\[ A^\top P_{\text{are}} + P_{\text{are}} A - K_{\text{are}}^\top R K_{\text{are}} + Q = 0 \]
\[ K_{\text{are}} = R^{-1} B^\top P_{\text{are}} \]
Suppose that we pick any stabilising $K$ and some $P \succ 0$ such that

$$\frac{d}{dt} [x(t)^\top P x(t)] \leq -x(t)^\top Q x(t) - u(t)^\top R u(t)$$

Since $K$ is stabilising, $\lim_{t \to \infty} x(t) = 0$. Integrate both sides above to get

$$x(0)^\top P x(0) \geq \int_0^\infty [x(\tau)^\top Q x(\tau) + u(\tau)^\top R u(\tau)] \, d\tau$$
$$\geq x(0)^\top P_{\text{are}} x(0)$$

The differential inequality condition above amounts to

$$(A - BK)^\top P + P(A - BK) \preceq -Q - K^\top R K, \quad P \succ 0$$

If this inequality is satisfied, then $K$ is stabilising and $P \succeq P_{\text{are}}$. 
Recall that $P \succeq P_{\text{are}}$ for every solution to our inequality. Find exact solution by solving:

$$\min_{(K,P)} \text{tr}(P)$$

subject to: 
\[
(A - BK)^\top P + P(A - BK) \preceq -Q - K^\top RK, \quad P \succ 0
\]

$P \succ 0$

Problem is nonlinear in $(P, K)$. Apply change of variables $S = P^{-1}$, $Y = -KP^{-1}$.

$$\max_{S,Y} \text{tr}(S)$$

subject to:
\[
\begin{bmatrix}
AS + SA^\top + Y^\top B^\top + BY & SQ^{1/2} & Y^\top R^{1/2} \\
Q^{1/2} S & -I & 0 \\
R^{1/2} Y & 0 & -I
\end{bmatrix} \preceq 0
\]

$S \succ 0$

Optimal controller is then $K^* = -Y^*(S^*)^{-1}$. 
Suppose that we want to bound the output energy

$$\int_{0}^{\infty} \left[ x(\tau)^{\top} Q x(\tau) + u(\tau)^{\top} R u(\tau) \right] \, d\tau \leq \gamma$$

starting from some specific initial condition $x(0) = x_0$.

This amounts to the constraint

$$x_0^{\top} P x_0 \leq \gamma \iff \begin{bmatrix} S & x_0 \\ x_0^{\top} & \gamma \end{bmatrix} \succeq 0, \ S := P^{-1}$$

where $x_0^{\top} P x_0$ bounds the integral from above.

NB: You can still never do better than $x_0^{\top} P \text{are} x_0$ as an upper bound.
Multi-objective analysis

Linear system with state feedback:

\[
\dot{x} = Ax + Bu \\
u = -Kx
\]

Stability Test

\text{Real}(\text{eig}(A - BK)) \leq [-\infty, 0] \iff \exists S_1 > 0, \quad (A - BK)S_1 + S_1(A - BK)^T < 0

Pole placement constraints

All CL poles with \(\theta(\lambda) < 50^\circ\) \iff \exists S_2 > 0, \quad L_\theta \otimes S_2 + M_\theta \otimes (AS_2) - M_\theta \otimes (BK S_2) + M_\theta^T \otimes (S_2 A^T) - M_\theta^T \otimes (S_2 K^T B^T) < 0

Integral cost constraints

\[\int_0^\infty \|u\|^2 \leq \gamma, \quad x(0) = x_0\] \iff \exists S_3 > 0, \quad \text{ARE-like inequalities satisfied}
Multi-objective design

Linear system with state feedback:

\[ \dot{x} = Ax + Bu \]
\[ u = -Kx \]

For synthesis we further constrain: \( S = S_1 = S_2 = S_3 > 0 \)

**Stability Test**

\[ \text{Real(eig}(A - BK)) \leq [-\infty, 0) \iff (A - BK)S + S(A - BK)^T < 0 \]

**Pole placement constraints**

All CL poles with \( \theta(\lambda) < 50^\circ \)

\[ \iff L_\theta \otimes S + M_\theta \otimes (AS) - M_\theta \otimes (BK)S \]
\[ + M_\theta^T \otimes (SA^T) - M_\theta^T \otimes (SK^T B^T) < 0 \]

**Integral cost constraints**

\[ \int_0^\infty \|u\|^2 \leq \gamma, \ x(0) = x_0 \iff \text{ARE-like inequalities satisfied using } (K, S) \]
Tuning / Matrix Weight Selection

The LQR optimal controller produces the minimum cost controller for some \((Q,R)\).

In practice, \((Q,R)\) are often just treated as ‘tuning knobs’. Some reasonable tuning rules:

- Choose \((Q, R)\) diagonal, with

\[
Q = \begin{bmatrix}
\frac{\alpha_1^2}{\|x_1\|_\infty^2} \\
\frac{\alpha_2^2}{\|x_2\|_\infty^2} \\
\vdots \\
\frac{\alpha_n^2}{\|x_n\|_\infty^2}
\end{bmatrix}, \quad R = \rho \begin{bmatrix}
\frac{\beta^2}{\|u_1\|_\infty^2} \\
\frac{\beta^2}{\|u_2\|_\infty^2} \\
\vdots \\
\frac{\beta^2}{\|u_m\|_\infty^2}
\end{bmatrix}
\]

- The terms \(\|x_i\|_\infty\) and \(\|u_i\|_\infty\) are peak magnitudes of each input and state.

- Constrain diagonal weights to \(\sum_i \alpha_i^2 = 1\) and \(\sum_i \beta_i^2 = 1\).

- Choose \(\rho \in (0, \infty)\) to balance relative importance of state and input penalties.