Exercise 6
LMIs for Systems Analysis and Design (Case Study III)

Sadegh Soudjani, Moritz Schulze Darup, and Dhruva Raman

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INTRODUCTION

In this exercise we will continue with the aircraft case study from earlier in the week. Properties such as stability, input-output gain and controllability will be studied using LMIs as the primary tool.

PROBLEMS

Consider again the aircraft model from Exercises 2 and 4 with the dynamics

\[
\dot{x} = \begin{pmatrix}
-0.003 & 0.039 & 0 & -0.322 \\
-0.065 & -0.319 & 7.747 & 0 \\
0.020 & -0.101 & -0.429 & 0 \\
0 & 0 & 1.000 & 0
\end{pmatrix} x + \begin{pmatrix}
0.01 & 1.00 \\
-0.18 & -0.04 \\
-1.16 & 0.60 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\Delta \delta_e \\
\Delta \delta_T
\end{pmatrix}
\] (1)

and the output

\[
y = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} x. 
\] (2)

Introducing the input vector \( u = (\Delta \delta_e, \Delta \delta_T)^T \), system (1)–(2) can be written in the standard form

\[
\dot{x} = Ax + Bu 
\] (3)

\[
y = Cx 
\] (4)

of linear time-invariant (LTI) systems.
1. Determine the asymptotic stability of the autonomous system $\dot{x} = Ax$ by constructing a quadratic Lyapunov function of the form $V(x) = x^TPx$. To do this set up an LMI with constraints

$$P > 0 \quad \text{and} \quad A^TP + PA < 0.$$ 

Note that these expressions are strict inequalities; how can you ensure that $P$ is positive definite rather than positive semidefinite?

2. When dealing with large-scale systems (e.g. flight formation, synchronization problems etc.) it is often desirable to impose some structure on the decision variables. With this in mind:
   a) Is it, for the given system, possible to construct a strictly diagonal Lyapunov function (i.e. impose the additional constraint $P = \text{diag}(p_{11}, \ldots, p_{44})$)?
   b) A second objective maybe to find a sparse Lyapunov function, i.e. a $P$ matrix with as many zero entries as possible. A heuristic for doing so is to minimise the 1-norm of $\text{vec}(P)$ and then apply a thresholding mask. Remember that the threshold may effect the definiteness of the matrix.

   **Attention:** Depending on the installed solvers, Yalmip might have problems to deal with the 1-norm objective.

3. The system (3)–(4) is said to be stabilizable (sometimes written as $(A,B)$ is stabilizable), if there exists a matrix $K$ such that $A - BK$ is stable. In the lectures earlier in the week it was shown how stabilizability can be determined by checking the rank of a matrix. Here we will determine this via an LMI. If there exists a $Q > 0$ and $\sigma > 0$ such that

$$AQ + QA^T + \sigma BB^T < 0$$

then the system is stabilizable and a stabilizing gain is given by $K = -(\sigma/2)B^TQ^{-1}$.

   a) Verify that the system is stabilizable. Construct $K$ and verify that the closed loop system is stable.
   b) A second approach to design a stabilizing gain is to find a controller matrix $K \in \mathbb{R}^{m \times n}$ and a $Q > 0$ such that

$$\begin{align*}
(A - BK)Q + Q(A - BK)^T &< 0. 
\end{align*}$$

Solve (5) by introducing an new variable $F$ to avoid nonlinear terms like $KQ$.

4. In the following, we will implement the SDP algorithm for computing an optimal $H_2$-norm state feedback controller. We first need to modify the aircraft model.

   a) The $B$ matrix defined at the top of the sheet will play the role of the $B_2$ matrix in the $H_2$ set up. Construct a random vector $B_1 \in \mathbb{R}^4$ to model the uncertain relation between $\dot{x}$ and $w$. 

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b) The output matrix $C_1$ will be $C$ and the $C_2$ matrix is $I_4$.

c) Set up the $D_{ij}$ matrices as appropriate. Make sure the dimensions are correct.

d) Implement the $H_2$ control design algorithm to minimize the $H_2$-norm of the closed loop.

**Hints:** The algorithm as it is in the lecture notes verifies that the $H_2$-norm of $\mathcal{S}(G, F)$ is less than 1. This is implemented via the trace constraint.

e) Verify your result by computing the $H_2$-norm of the controlled system using the MATLAB function `norm`.

5. We saw in the lecture how the *gain* of a system could be measured by a suitable norm, i.e. the $H_2$-norm. A (possibly) more intuitive choice is the $H_\infty$-norm which can be computed via a storage function argument with quadratic supply-rate and thus optimised using an SDP.

a) Consider the quadratic storage function $V(x) = x^T P x$ with $P \succ 0$ and the supply rate is

$$w(u, y) = \gamma^2 u^T u - y^T y.$$ 

Without proof, we claim that satisfaction of the dissipation inequality, i.e.,

$$\dot{V}(x) \leq w(u, y) \quad (6)$$

implies $\|G\|_{H_\infty} < \gamma$. Rewrite (6) in terms of an LMI.

**Hints:** Calculate $\dot{V}(x)$ for the open system. Then rewrite (6) in the form

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0$$

and introduce the auxiliary variable $\rho = \gamma^2$.

b) Solve the SDP by minimizing $\rho$ subject to the LMI constraint.

c) Verify your result by computing the $H_\infty$-norm of the open system using the MATLAB function `norm`. 

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