Exercise 5
A brief introduction to LMIs

Sadegh Soudjani, Moritz Schulze Darup, and Dhruva Raman

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INTRODUCTION

Many problems in control theory can be formulated and solved by taking linear matrix inequalities (LMIs) into account. The exercise consequently deals with the formulation and numerical solution of LMIs.

PROBLEMS

1. Consider the following optimization problem

minimize \( x_1^2 + x_2^2 \)

subject to

\[ f_1(x) = \frac{x_1}{1 + x_2^2} \leq 0 \]

\[ h_1(x) = (x_1 + x_2)^2 = 0. \]

with decision variables \( x_1 \) and \( x_2 \).

a) Explain why this is not a convex program according to the strict definition given in the lecture (see also [1, p. 136]).

b) Determine (by hand) what the feasible set of this program is, i.e. the set \( \{ x \in \mathbb{R}^2 \mid f_1(x) \leq 0, \ h_1(x) = 0 \} \) and use it to write down an equivalent but convex problem. What is the optimal solution?

2. As we saw in the lecture this morning, the following quadratic matrix inequality is frequently encountered when designing controllers:

\[ A^T P + PA + PBR^{-1}B^T P + Q \prec 0 \quad (1) \]

where \( P \in \mathbb{S}_{++}^n \) is the decision variable, and \( Q \in \mathbb{S}_+^n \), \( R \in \mathbb{S}_{++}^n \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) are given.
a) Use the Schur complement procedure to write (1) as a linear matrix inequality.

b) Use the MATLAB function \texttt{rss} to generate a random stable state-space system in terms of the matrices $A$, $B$, $C$, and $D$. Solve the LMI derived in the previous task by additionally taking the constraint $P \succeq 0$ into account.

c) Add a trace minimisation objective function.

3. We next consider a geometric problem and solve it using semidefinite programming. Suppose we are given $k$ ellipsoids $E_1, \ldots, E_k$ and the challenge is to find the smallest ball that contains all $k$ ellipsoids. Below are a set of hints and MATLAB pointers that will help you achieve this goal.

**Hints:** An ellipsoid can be represented by a level-set of a quadratic function of the form

$$e(x) = x^T Ax + 2b^T x + c, \quad A \succ 0.$$  

(2)

The ellipsoid is thus the set $E = \{x \in \mathbb{R}^n \mid e(x) \leq 0\}$. MATLAB’s symbolic engine makes it easy to plot level-sets of a function. For example, the code

\begin{verbatim}
> syms x1 x2
> x = [x1 x2].'; P = diag([2,3]);
> ezplot(x.'*P*x-4)
\end{verbatim}

plots the level-curve of the ellipses $x^T P x = 4$. Suppose we have two ellipsoids $E$ and $\hat{E}$ which we represent by the functions

$$f(x) = x^T Ax + 2b^T x + c \quad \text{and} \quad \hat{f}(x) = x^T \hat{A} x + 2\hat{b}^T x + \hat{c},$$

then $E$ contains $\hat{E}$ if and only if there exists a $\tau > 0$ such that

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \preceq \tau \begin{bmatrix} \hat{A} & \hat{b} \\ \hat{b}^T & \hat{c} \end{bmatrix}.$$ 

(3)

Using these hints, the following tasks should be solved.

a) Construct three ellipsoids $E_1, \ldots, E_3 \subset \mathbb{R}^2$ (i.e., ellipses) and plot them (e.g., using \texttt{ezplot}). Make sure $A_1, \ldots, A_3 \in \mathbb{R}^{2 \times 2}$ are positive def. and symmetric.

b) The ball $B$ we wish to find can also be expressed as an ellipsoid of the form (2) with $A_0 = I_2$, $b_0 = -\mu$, and $c_0 = \gamma$. Thereby, $\mu$ denotes the center of the ball. Show that the radius of the ball is given by $r = \sqrt{\mu^T \mu - \gamma}$.

c) Using (3), the smallest ball containing the three ellipses can be computed by solving a semidefinite program. To this end, we introduce the decision variables $\mu$ and $\gamma$ and derive the LMI constraints guaranteeing $E_i \subseteq B$. We then solve the LMI while minimizing the quadratic objective function $\mu^T \mu - \gamma$ to obtain the smallest radius.

d) Plot the resulting ball together with all ellipses to verify the solution.
e) Assume we want to consider a linear objective function. In this case, we can introduce the auxiliary decision variable $\rho$ and minimize $\rho$ subject to the new constraint

$$\mu^T \mu - \gamma \leq \rho \tag{4}$$

and the LMI constraints from 3.c). Clearly, (4) is quadratic in $\mu$. Try to rewrite (4) as an LMI using the Schur compliment. Solve the rewritten optimization problem and check whether the solution is identical to 3.c).

f) **Extra credit:** Solve the task again for three ellipsoids in $\mathbb{R}^3$.

**Hints:** the function `ezsurf` can be used for 3-dimensional plotting.

**REFERENCES**