Exercise 3
Linearization and Lyapunov Stability

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November 24, 2015

INTRODUCTION

When a system is unstable, the output of the system may be infinite even though the
input to the system was finite. This causes possible physical damage or dangerous be-
havior by the system, which must be avoided. Before designing controllers that stabilize
the system, it is first important to understand what stability is and how it is determined.
The exercise deals with the study of stability in relation with Lyapunov equation, linear
transformation of systems, and linearization of nonlinear systems.

PROBLEMS

1. Assume there exists a pair \((P, Q)\) that satisfies the Lyapunov equation

\[ A^T P + PA - Q = 0, \]

for a given matrix \(A \in \mathbb{R}^{n \times n}\). Compute a pair of matrices \((\tilde{P}, \tilde{Q})\) satisfying

\[ \tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} - \tilde{Q} = 0, \]

in terms of \((P, Q)\), where \(\tilde{A} = T^{-1}AT\).

2. The characteristic polynomial of any matrix \(A\) is \(p(s) = \det(sI - A)\). Show that
the characteristic polynomials of \(A\) and \(\tilde{A} = T^{-1}AT\) are the same.

**Hints:** Use that \(\det(XY) = \det(X) \det(Y)\).

3. Compute the equilibrium points of the predator-prey model in terms of parameters
\((a, b, c, d)\)

\[
\begin{align*}
\dot{h}(t) &= bh(t) - a\ell(t)h(t) \\
\dot{\ell}(t) &= c\ell(t)h(t) - d\ell(t).
\end{align*}
\]
Linearize the system around the equilibrium points and compute the eigenvalues of the linearized system around each equilibrium point.

Use the MATLAB function \texttt{quiver} to draw the vector field of

a) the nonlinear system (1).

b) the linear system approximation around the equilibrium point \(x_0 = \begin{bmatrix} d/c & b/a \end{bmatrix}^T\).

c) the approximation error \(e(x) = f(x) - Ax\).

4. Consider the linearized model of the inverted pendulum discussed in Exercise (2),

\[
\dot{x}(t) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & \frac{mg}{M} & 0 & 0 \\
0 & \frac{(M+m)g}{M} & 0 & 0 \\
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
0 \\
\frac{1}{M} \\
\frac{1}{Ml} \\
\end{bmatrix} u(t),
\]

with parameters \(M = 4\), \(m = 1\), \(l = 1\), \(g = 9.81\).

a) Compute the eigenvalues of matrix \(A\). Select \(Q = I\) and try solving the Lyapunov equation associated to the system. What can be inferred about the asymptotic stability of the equilibrium \(q = 0\)?

b) Consider the input \(u(t) = -Kx(t) = \begin{bmatrix} -1 & 80 \\ -110 & 0 \end{bmatrix} x(t)\) as a state feedback to the system. Write down the closed-loop system as

\[
\dot{x}(t) = Ax(t) + Bu(t) = (A - BK)x(t),
\]

and obtain a solution to the Lyapunov equation with \(Q = I\) and \(\tilde{A} = (A - BK)\) corresponding to the closed-loop system (2). Are the solutions to the closed-loop linear system (2) asymptotically stable?