Exercise 1
Modelling and Simulation

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INTRODUCTION

The design of a suitable controller for a given plant requires a rough understanding of the system’s behavior. Usually, we use mathematical models to describe and to simulate the system dynamics. The exercise addresses the derivation of such models and their simulative analysis.

PROBLEMS

1. Derive the (nonlinear) model for the inverted pendulum on a moving cart discussed in the lecture this morning (see Fig. 1). Write the model equations in the form

\[ \mathcal{M}(q) \ddot{q} + \mathcal{K}(q, \dot{q}) = \mathcal{R} F. \]

with \( \mathcal{M} \in \mathbb{R}^{2 \times 2} \) and \( \mathcal{K}, \mathcal{R} \in \mathbb{R}^2 \).

Figure 1: Inverted pendulum on a cart.
**Hints:** Derive the kinetic energy $T$ and the potential energy $V$ to define the Lagrangian

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

and use the Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = U$$

with $q = \begin{pmatrix} p \\ \theta \end{pmatrix}$ and $U = \begin{pmatrix} F \\ 0 \end{pmatrix}$

to obtain the equations of motion.

2. Using MATLAB, compute trajectories for the predator-prey model

$$\dot{h}(t) = bh(t) - ah(t) \ell(t)$$
$$\dot{\ell}(t) = ch(t) \ell(t) - d \ell(t)$$

with $a = 5$, $b = 5$, $c = 4$, and $d = 8$. Plot trajectories as functions of time and on the phase plane for the following initial conditions

$$x_0^{(1)} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad x_0^{(2)} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \quad x_0^{(3)} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \quad \text{and} \quad x_0^{(4)} = \begin{pmatrix} d/c \\ b/a \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and explain the results (related to the model equations).

**Hints:** Use the MATLAB function ode23.

3. Using MATLAB, compute trajectories for the mass-spring-damper system described by the ordinary differential equation (ODE)

$$m \ddot{q}(t) = -c \dot{q}(t) - k q(t) + u(t)$$

with $m = 1$, $c = 1.5$, and $k = 2$. Thereby, assume the input $u(t)$ is a square wave of period $\frac{\pi}{2}$ and amplitude 5. Simulate the system behavior for 100 s and plot trajectories as functions of time and on the phase plane with $x = (q(t), \dot{q}(t))^T$.

**Hints:** Use the MATLAB functions ss, square, and lsim.