For airplanes, the reference area $S$ is taken as the wing platform area and the characteristic length $l$ is taken as the wing span for the rolling and yawing moment and the mean chord for the pitching moment. For rockets and missiles, the reference area is usually taken as the maximum cross-sectional area, and the characteristic length is taken as the maximum diameter.

The aerodynamic coefficients $C_x$, $C_y$, $C_z$, $C_m$, and $C_n$ primarily are a function of the Mach number, Reynolds number, angle of attack, and sideslip angle; they are secondary functions of the time rate of change of angle of attack and sideslip, and the angular velocity of the airplane.

The aerodynamic force and moment acting on the airplane and its angular and translational velocity are illustrated in Figure 1.10. The $x$ and $z$ axes are in the plane of symmetry, with the $x$ axis pointing along the fuselage and the positive $y$ axis along the right wing. The resultant force and moment, as well as the airplane's velocity, can be resolved along these axes.

The angle of attack and sideslip can be defined in terms of the velocity components as illustrated in Figure 1.11. The equations for $\alpha$ and $\beta$ follow:

$$\alpha = \tan^{-1} \frac{V_y}{u}$$

(1.67)

and

$$\beta = \sin^{-1} \frac{V_z}{V}$$

(1.68)

where

$$V = (u^2 + v^2 + w^2)^{1/2}$$

(1.69)

If the angle of attack and sideslip are small, that is, $< 15^\circ$, then Equations (1.67)
FIGURE 3.3
Relationship between body and inertial axes systems.

Integration of these equations yields the airplane's position relative to the fixed frame of reference.

The relationship between the angular velocities in the body frame \((p, q, \text{ and } r)\) and the Euler rates \((\psi, \theta, \text{ and } \Phi)\) also can be determined from Figure 3.3:

\[
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -S_\theta \\
0 & C_\theta & C_\phi S_\theta \\
0 & -S_\phi & C_\phi C_\theta
\end{bmatrix}
\begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\Phi}
\end{bmatrix}
\tag{3.31}
\]

Equation (3.31) can be solved for the Euler rates in terms of the body angular velocities:

\[
\begin{bmatrix}
\dot{\Phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
1 & S_\phi \tan \theta & C_\phi \tan \theta \\
0 & C_\phi & -S_\phi \\
0 & S_\phi \sec \theta & C_\phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\tag{3.32}
\]

By integrating these equations, one can determine the Euler angles \(\psi, \theta, \text{ and } \Phi\).

3.4 GRAVITATIONAL AND THRUST FORCES

The gravitational force acting on the airplane acts through the center of gravity of the airplane. Because the body axis system is fixed to the center of gravity, the gravitational force will not produce any moments. It will contribute to the external force acting on the airplane, however, and have components along the respective body axes. Figure 3.4 shows that the gravitational force components acting along the body axis are a function of the airplane's orientation in space. The gravitational

FIGURE 3.4
Components of gravitational force acting along the body axis.
force components along the x, y, and z axes can be easily shown to be

\[ (F_x)_{\text{gravity}} = -mg \sin \theta \]
\[ (F_y)_{\text{gravity}} = mg \cos \theta \sin \Phi \]
\[ (F_z)_{\text{gravity}} = mg \cos \theta \cos \Phi \]  

(3.33)

The thrust force due to the propulsion system can have components that act along each of the body axis directions. In addition, the propulsive forces also can create moments if the thrust does not act through the center of gravity. Figure 3.5 shows some examples of moments created by the propulsive system.

The propulsive forces and moments acting along the body axis system are denoted as follows:

\[ (F_x)_{\text{propulsive}} = X_T \quad (F_y)_{\text{propulsive}} = Y_T \quad (F_z)_{\text{propulsive}} = Z_T \]  

(3.34)

and

\[ (L)_{\text{propulsive}} = L_T \quad (M)_{\text{propulsive}} = M_T \quad (N)_{\text{propulsive}} = N_T \]  

(3.35)

Table 3.1 gives a summary of the rigid body equations of motion.

### 3.5 SMALL-DISTURBANCE THEORY

The equations developed in the previous section can be linearized using the small-angle approximation. The motion of the airplane consists of small deviations about a steady flight condition. Obviously, this theory cannot be applied to problems in which large-amplitude motions are to be expected (e.g., spinning or stalled flight). However, in many cases the small-disturbance theory yields sufficient accuracy for practical engineering purposes.

All the variables in the equations of motion are replaced by a reference value plus a perturbation or disturbance:

\[ u = u_0 + \Delta u \quad v = v_0 + \Delta v \quad w = w_0 + \Delta w \]
\[ p = p_0 + \Delta p \quad q = q_0 + \Delta q \quad r = r_0 + \Delta r \]
\[ X = X_0 + \Delta X \quad Y = Y_0 + \Delta Y \quad Z = Z_0 + \Delta Z \]
\[ M = M_0 + \Delta M \quad N = N_0 + \Delta N \quad L = L_0 + \Delta L \]
\[ \delta = \delta_0 + \Delta \delta \]  

(3.36)

For convenience, the reference flight condition is assumed to be symmetric and the propulsive forces are assumed to remain constant. This implies that

\[ v_0 = p_0 = q_0 = r_0 = \Phi_0 = \Psi_0 = 0 \]  

(3.37)

Furthermore, if we initially align the x axis so that it is along the direction of the initial velocity component, the

\[ \Phi = \Psi = 0 \]

(3.38)

Then the aerodynamic forces include only those components that are perturbations of the reference condition.
Substituting the expression for $\Delta X$ into the force equation yields:

$$
\frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial \delta_e} \Delta \delta_e + \frac{\partial X}{\partial \delta_r} \Delta \delta_r - mg \Delta \theta \cos \theta_0 = m \Delta u
$$

or on rearranging

$$
\left( m \frac{d}{dt} - \frac{\partial X}{\partial u} \right) \Delta u - \left( \frac{\partial X}{\partial w} \Delta w + (mg \cos \theta_0) \Delta \theta \right) \Delta \delta_e + \frac{\partial X}{\partial \delta_e} \Delta \delta_r
$$

The equation can be rewritten in a more convenient form by dividing through by the mass $m$:

$$
\left( \frac{d}{dt} - X_u \right) \Delta u - X_w \Delta w + (g \cos \theta_0) \Delta \theta = X_{\delta_e} \Delta \delta_e + X_{\delta_r} \Delta \delta_r
$$

where $X_u = \partial X/\partial u/m$, $X_w = \partial X/\partial w/m$, and so on are aerodynamic derivatives divided by the airplane's mass.

The change in aerodynamic forces and moments are functions of the motion variables $\Delta u$, $\Delta w$, and so forth. The aerodynamic derivatives usually the most important for conventional airplane motion analysis follow:

$$
\begin{align*}
\Delta X &= \frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial \delta_e} \Delta \delta_e + \frac{\partial X}{\partial \delta_r} \Delta \delta_r \\
\Delta Y &= \frac{\partial Y}{\partial v} \Delta v + \frac{\partial Y}{\partial \delta_e} \Delta \delta_e + \frac{\partial Y}{\partial \delta_r} \Delta \delta_r \\
\Delta Z &= \frac{\partial Z}{\partial u} \Delta u + \frac{\partial Z}{\partial w} \Delta w + \frac{\partial Z}{\partial \delta_e} \Delta \delta_e + \frac{\partial Z}{\partial \delta_r} \Delta \delta_r \\
\Delta L &= \frac{\partial L}{\partial \delta_e} \Delta \delta_e + \frac{\partial L}{\partial \delta_r} \Delta \delta_r \\
\Delta M &= \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial \delta_e} \Delta \delta_e + \frac{\partial M}{\partial \delta_r} \Delta \delta_r \\
\Delta N &= \frac{\partial N}{\partial \delta_e} \Delta \delta_e + \frac{\partial N}{\partial \delta_r} \Delta \delta_r
\end{align*}
$$

The aerodynamic forces and moments can be expressed as a function of all the motion variables; however, in these equations only the terms that are usually significant have been retained. Note also that the longitudinal aerodynamic control surface was assumed to be an elevator. For aircraft that use either a canard or all-moveable stabilizer, the control term would be replaced by
TABLE 3.2
The linearized small-disturbance longitudinal and lateral rigid body equations of motion

<table>
<thead>
<tr>
<th>Longitudinal equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dt} - \dot{X}<em>c ) ( \Delta u ) - ( X_c \Delta \omega ) + ( (g \cos \theta_0) \Delta \dot{\theta} ) = ( X</em>\Delta \Delta \delta_\dot{\theta} + X_{\Delta \varepsilon} \Delta \delta_r )</td>
</tr>
<tr>
<td>(-Z_c \Delta u + \left( \frac{1 - Z_c}{d} - Z_c \right) \Delta \omega - \left( u_0 + Z_c \frac{d}{dt} - g \sin \theta_0 \right) \Delta \dot{\theta} = Z_\Delta \Delta \delta_r + Z_{\Delta \varepsilon} \Delta \delta_r )</td>
</tr>
<tr>
<td>(-M_{\Delta u} - \left( M_{\Delta \varepsilon} \frac{d}{dt} + M_\varepsilon \right) \Delta \omega + \left( \frac{d^2}{dt^2} - M_{\Delta \varepsilon} \frac{d}{dt} \right) \Delta \dot{\theta} = M_\Delta \Delta \delta_r + M_{\Delta \varepsilon} \Delta \delta_r )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lateral equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dt} - \dot{Y}<em>c ) ( \Delta r ) - ( Y_c \Delta \delta_p ) + ( (u_0 - Y_c) \Delta \dot{r} ) - ( g \cos \theta_0 \Delta \psi ) = ( Y</em>\Delta \Delta \delta_r )</td>
</tr>
<tr>
<td>(-L_c \Delta r + \left( \frac{d}{dt} - L_\varepsilon \right) \Delta \delta_p - \left( \frac{L_\varepsilon}{I_{\Delta \varepsilon}} \frac{d}{dt} + L_\varepsilon \right) \Delta \dot{r} = L_\Delta \Delta \delta_r + L_{\Delta \varepsilon} \Delta \delta_r )</td>
</tr>
<tr>
<td>(-N_c \Delta r - \left( \frac{L_\varepsilon}{I_{\Delta \varepsilon}} \frac{d}{dt} + N_\varepsilon \right) \Delta \delta_p - \left( \frac{d}{dt} - N_\varepsilon \right) \Delta \dot{r} = N_\Delta \Delta \delta_r + N_{\Delta \varepsilon} \Delta \delta_r )</td>
</tr>
</tbody>
</table>

3.6 AERODYNAMIC FORCE AND MOMENT REPRESENTATION

In previous sections we represented the aerodynamic force and moment contributions by means of the aerodynamic stability coefficients. We did this without explaining the rationale behind the approach.

The method of representing the aerodynamic forces and moments by stability coefficients was first introduced by Bryan over three-quarters of a century ago [3.1, 3.3]. The technique proposed by Bryan assumes that the aerodynamic forces and moments can be expressed as a function of the instantaneous values of the perturbation variables. The perturbation variables are the instantaneous changes from the reference conditions of the translational velocities, angular velocities, control deflections, and their derivatives. With this assumption, we can express the aerodynamic forces and moments by means of a Taylor series expansion of the perturbation variables about the reference equilibrium condition. For example, the change in the force in the x direction can be expressed as follows:

\[
\Delta X(u, u, w, \dot{u}, \delta_\varepsilon, \dot{\delta}_r)
= \frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial \dot{u}} \Delta \dot{u} + \cdots + \frac{\partial X}{\partial \delta_\varepsilon} \Delta \delta_\varepsilon + \text{H.O.T. (higher order terms)}
\]  

(3.50)

The term \( \partial X/\partial u \), called the stability derivative, is evaluated at the reference flight condition.

Aerodynamic Force and Moment Representation

coefficient \( C_{\Delta \varepsilon} \) as follows:

\[
\frac{\partial X}{\partial u} = C_{\Delta \varepsilon} \frac{1}{u_0} QS
\]

(3.51)

where

\[
C_{\Delta \varepsilon} = -\frac{\partial C_{\Delta \varepsilon}}{\partial u/\varepsilon_0}
\]

(3.52)

Note that the stability derivative has dimensions, whereas the stability coefficient is defined so that it is nondimensional.

The preceding discussion may seem as though we are making the aerodynamic force and moment representation extremely complicated. However, by assuming that the perturbations are small we need to retain only the linear terms in Equation (3.50). Even though we have retained only the linear terms, the expressions still may include numerous first-order terms. Fortunately, many of these terms also can be neglected because their contribution to a particular force or moment is negligible. For example, we have examined the pitching moment in detail in Chapter 2. If we express the pitching moment in terms of the perturbation variables, as indicated next,

\[
M(u, v, w, \dot{u}, \dot{v}, \dot{w}, p, q, r, \delta_\varepsilon, \varepsilon, \dot{\varepsilon}, \dot{\delta}_r, \dot{\delta}_r)
= \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial \dot{u}} \Delta \dot{u} + \frac{\partial M}{\partial \delta_p} \Delta \delta_p + \cdots + \frac{\partial M}{\partial \delta_r} \Delta \delta_r + \cdots
\]

(3.53)

it should be quite obvious that terms such as \( (\partial M/\partial \varepsilon) \Delta \varepsilon \) and \( (\partial M/\partial \delta_p) \Delta \delta_p \) are not going to be significant for an airplane. Therefore, we can neglect these terms in our analysis.

In the following sections, we shall use the stability derivative approach to represent the aerodynamic forces and moments acting on the airplane. The expressions developed for each of the forces and moments will include only the terms usually important in studying the airplane’s motion. The remaining portion of this chapter is devoted to presentation of methods for predicting the longitudinal and lateral stability coefficients. We will confine our discussion to methods that are applicable to subsonic flight speeds. Note that many of the stability coefficients vary significantly with the Mach number. This can be seen by examining the data on the A-4D airplane in Appendix B or by examining Figure 3.6.

We have developed a number of relationships for estimating the various stability coefficients; for example, expressions for some of the static stability coefficients such as \( C_{\Delta \varepsilon} \), \( C_{\Delta \varepsilon} \), and \( C_{\Delta \varepsilon} \) were formulated in Chapter 2. Developing prediction methods for all of the stability derivatives necessary for performing vehicle motion analysis would be beyond the scope of this book. Therefore, we shall confine our attention to the development of several important dynamic derivatives and simply refer the reader to the US Air Force Stability and Control DATCOM [3.4].
TABLE 3.5
Summary of longitudinal derivatives

\[
X_a = \frac{-C_{D_a} + 2C_{D_b}}{m u_0} QS (s^{-1}) \quad \quad X_w = \frac{-(C_{D_a} - C_{D_b})}{m u_0} QS (s^{-1})
\]

\[
Z_a = \frac{-(C_{D_a} + 2C_{D_b})}{m u_0} QS (s^{-1})
\]

\[
Z_w = \frac{-(C_{D_a} - C_{D_b})}{m u_0} QS (s^{-1})
\]

\[
Z_a = u_0 Z_w (t/s^2) \text{ or (m/s)} \quad Z_w = u_0 Z_a (t/s) \text{ or (m/s)}
\]

\[
Z_v = -C_{D_a} \frac{c}{2 u_0} QS/m (t/s) \text{ or (m/s)} \quad Z_k = -C_{D_a} QS/m (t/s)
\]

\[
M_a = C_{m_a} \left( \frac{Q S C}{u_0 l_e} \right) \left( \frac{1}{R \cdot s} \text{ or } \frac{1}{m \cdot s} \right)
\]

\[
M_w = C_{m_w} \left( \frac{Q S C}{u_0 l_e} \right) \left( \frac{1}{R \cdot s} \text{ or } \frac{1}{m \cdot s} \right)
\]

\[
M_a = C_{m_a} \frac{c}{2 u_0} Q S C (f t)^{-1} \quad M_w = C_{m_w} Q S C (f t)^{-1}
\]

\[
M_a = u_0 M_w (s^{-2}) \quad M_w = u_0 M_a (s^{-2})
\]

\[
M_a = C_{m_a} \left( \frac{Q S C}{u_0 l_e} \right) (s^{-1}) \quad M_w = C_{m_w} \left( \frac{Q S C}{u_0 l_e} \right) (s^{-1})
\]

TABLE 3.6
Summary of lateral directional derivatives

\[
Y_a = \frac{Q S C_{a b}}{m} (t/s) \text{ or (m/s)} \quad N_a = \frac{Q S b C_{a b}}{l_e} (s^{-2}) \quad L_a = \frac{Q S b C_{a b}}{l_e} (s^{-2})
\]

\[
Y_a = \frac{Q S b C_{a b}}{2 m u_0} (t/s) \quad N_a = \frac{Q S b C_{a b}}{2 l_e u_0} (s^{-1}) \quad L_a = \frac{Q S b C_{a b}}{2 l_e u_0} (s^{-1})
\]

\[
Y_w = \frac{Q S C_{a b}}{m} (t/s) \text{ or (m/s)} \quad Y_b = \frac{Q S C_{a b}}{m} (t/s) \text{ or (m/s)}
\]

\[
N_a = \frac{Q S b C_{a b}}{l_e} (s^{-2}) \quad N_b = \frac{Q S b C_{a b}}{l_e} (s^{-2})
\]

\[
L_a = \frac{Q S b C_{a b}}{l_e} (s^{-2}) \quad L_b = \frac{Q S b C_{a b}}{l_e} (s^{-2})
\]

As noted earlier, there are many more derivatives for which we could develop prediction methods. The few simple examples presented here should give the reader an appreciation of how one would go about determining estimates of the aerodynamic stability coefficients. A summary of some of the theoretical predic-
embarrassment, one must always verify that the assumptions used in developing the equations one wishes to use are consistent with the problem one is attempting to solve. This is particularly important when solving problems related to aircraft dynamics.

In the following sections we shall examine the longitudinal motion of an airplane without control input. The longitudinal motion of an airplane (controls fixed) disturbed from its equilibrium flight condition is characterized by two oscillatory modes of motion. Figure 4.10 illustrates these basic modes. We see that one mode is lightly damped and has a long period. This motion is called the long-period or phugoid mode. The second basic motion is heavily damped and has a very short period; it is appropriately called the short-period mode.

### 4.4.1 State Variable Representation of the Equations of Motion

The linearized longitudinal equations developed in Chapter 3 are simple, ordinary linear differential equations with constant coefficients. The coefficients in the differential equations are made up of the aerodynamic stability derivatives, mass, and inertia characteristics of the airplane. These equations can be written as a set of first-order differential equations, called the state-space or state variable equations and represented mathematically as

\[
\dot{x} = Ax + B\eta
\]  

(4.49)

where \(x\) is the state vector, \(\eta\) is the control vector, and the matrices \(A\) and \(B\) contain the coefficients of the system.

The linearized longitudinal set of equations developed earlier are repeated here:

\[
\left( \frac{d}{dt} - X_u \right) \Delta u - X_w \Delta w + (g \cos \theta_0) \Delta \theta = X_\delta \Delta \delta + X_{\delta r} \Delta \delta_r
\]

\[
-Z_u \Delta u + \left[ (1 - Z_w) \frac{d}{dt} - Z_w \right] \Delta w - \left[ (u_0 + Z_\delta) \frac{d}{dt} - g \sin \theta_0 \right] \Delta \theta = Z_\delta \Delta \delta + Z_{\delta r} \Delta \delta_r
\]

(4.50)

\[
-Z_u \Delta u - \left( M_u \frac{d}{dt} + M_w \right) \Delta w + \left( \frac{d^2}{dt^2} - M_{\delta \dot{u}} \frac{d}{dt} \right) \Delta \theta = M_\delta \Delta \delta + M_{\delta r} \Delta \delta_r
\]

where \(\Delta \delta\) and \(\Delta \delta_r\) are the aerodynamic and propulsive controls, respectively.

In practice, the force derivatives \(Z_u\) and \(Z_w\) are usually neglected because they contribute very little to the aircraft response. Therefore, to simplify our presentation of the equations of motion in the state-space form we will neglect both \(Z_u\) and \(Z_w\). Rewriting the equations in the state-space form yields

\[
\begin{bmatrix}
\Delta \dot{u} \\
\Delta \dot{w} \\
\Delta \dot{q} \\
\Delta \dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
X_u & X_w & 0 & -g \\
Z_u & Z_w & u_0 & 0 \\
0 & 0 & 1 & 0 \\
M_\delta + M_u Z_u & M_w + M_u Z_w & M_\delta + M_u u_0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta w \\
\Delta q \\
\Delta \theta
\end{bmatrix}
\]

(4.51)

and the matrices \(A\) and \(B\) are given by

\[
A =
\begin{bmatrix}
X_u & X_w & 0 & -g \\
Z_u & Z_w & u_0 & 0 \\
0 & 0 & 1 & 0 \\
M_\delta + M_u Z_u & M_w + M_u Z_w & M_\delta + M_u u_0 & 0
\end{bmatrix}
\]

(4.53)

\[
B =
\begin{bmatrix}
X_\delta & X_{\delta r} \\
Z_\delta & Z_{\delta r} \\
M_\delta + M_u Z_u & M_w + M_u Z_w \\
M_\delta + M_u Z_u & M_w + M_u Z_w
\end{bmatrix}
\]

(4.54)