Lecture 3

• Continuation of theory of linear ODE

• Equilibria, Stability (a-la-Lyapunov, hints to BIBO)

• Lyapunov techniques for stability analysis
Qualitative Analysis of ODE: phase portraits

- Consider planar case, $x \in \mathbb{R}^2$ (dominant mode)

- By studying properties of vector field, build *phase portrait*
Qualitative Analysis of ODE: Stability

• Many different “kinds.” Most relevant for this class
  – a-la-Lyapunov
  – BIBO (bounded input, bounded output)

• $x_e$ is an equilibrium point for $\dot{x} = f(x)$ if $f(x_e) = 0$

• Simple case of linear ODE: origin (at least) is an equilibrium point

• Consider $x(t, a), t \geq 0$, a solution of $\dot{x} = f(x) : x(0, a) = a$

• An equilibrium point $x_e$ is stable (a-la-Lyapunov) if
  \[
  \forall \epsilon > 0, \exists \delta > 0 : \|a - x_e\| < \delta \rightarrow \|x(t, a) - x_e\| < \epsilon, \forall t \geq 0
  \]
Qualitative Analysis of ODE: Asymptotic Stability

• An equilibrium point $x_e$ is asymptotically stable if
  1. it is stable (a-la-Lyapunov)
  2. $x(t, a) \rightarrow x_e$, as $t \rightarrow \infty$

• pay attention to: local vs global validity of the above notions
  (e.g., can quantify domains of attraction)

• in the linear ODE case, recall solution form, depending on matrix exponential

• intuitively, matrix eigenvalues play a role
Lyapunov Stability – Simple Linear Planar Examples

\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= x
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= -0.1x - y \\
\dot{y} &= x
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= -0.1x - y \\
\dot{y} &= x + y
\end{align*}
\]

\[
\lambda_{1,2} = 0 \pm i \\
\lambda_{1,2} = -0.05 \pm i0.99 \\
\lambda_{1,2} = 0.45 \pm i0.84
\]
Stability of Linear ODE

- For LTI system $\dot{x} = Ax, x \in \mathbb{R}^n$, stability of equilibrium is related to the eigenvalues of state matrix $A$

- $\lambda(A) = \{ s \in \mathbb{C} : \det(sI - A) = 0 \}$

- $\det(sI - A)$ is known as the characteristic polynomial

- diagonal case: $A = \text{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_2 & \ddots \\ 0 & \cdots & \lambda_n \end{pmatrix}$
• Result: equilibrium point $x_e = 0$ is
  
  – stable if $Re(\lambda_i) \leq 0$,
  
  – asymptotically stable if $Re(\lambda_i) < 0, i = 1, \ldots, n$

• case $A$ diagonalizable: $A = T^{-1}\Lambda T$ (invertible matrix $T$ provides similarity transformation) → study stability on matrix $\Lambda$

\[ \det(sI - A) = \det(sI - \Lambda) \]
Linearization around Equilibrium Point

• Consider autonomous nonlinear ODE: \( \dot{x}(t) = f(x(t)), x(0) = x_0, x \in \mathbb{R}^n \)

• Set of equilibria: \( \{ z \in \mathbb{R}^n : f(z) = 0 \} \)

• Jacobian of vector field \( f \) at point \( x \):

\[
J(x) : [J(x)]_{i,j} = \left. \frac{\partial f_i}{\partial x_j} \right|_x
\]

• Of particular interest: linearization of a ODE around an equilibrium point \( x_e \in \mathbb{R}^n \)

\[
\dot{x}(t) = J(x(t)), \quad x(0) = x_0 \in N(x_e, \delta)
\]
• **Example:** inverted pendulum (balance system, Segway)

• Original Model (no cart, no input)  
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= \sin x_1 - cx_2
  \end{align*}
  \]

• Linearization around origin  
  \[
  \begin{align*}
  \dot{z}_1 &= z_2 \\
  \dot{z}_2 &= \sin x_1 - cx_2
  \end{align*}
  \]

- locality of above approximation \(\rightarrow\) can study stability only “locally”!
Lyapunov Stability Analysis of ODE

• Let $x \in \mathbb{R}^n$. A function $V(x)$ is called positive (semi)definite in $\mathbb{R}^n$ if $V(x) > 0$ ($V(x) \geq 0$) for all $x \in \mathbb{R}^n$ with $x \neq 0$, and $V(0) = 0$

• Examples: ($x \in \mathbb{R}^2$)

\[
\begin{align*}
 & x_1^2 + 2x_2^2 & \text{positive definite} \\
 & (x_1 + x_2)^2 & \text{positive semidefinite} \\
 & -x_1^2 - (3x_1 + 2x_2)^2 & \text{negative definite} \\
 & x_1x_2 + x_2^2 & \text{indefinite} \\
 & x_1^2 + \frac{2x_2^2}{1+x_2^2} & \text{positive definite}
\end{align*}
\]

• Special case: $V(x) := x^T P x$, where $P = P^T \in \mathbb{R}^{n \times n}$ ($P$ is symmetric). Then $V > 0 \iff P > 0$ (pos. def.) $\iff \forall$ eigenvalues $Re(\lambda_i(P)) > 0$
Lyapunov Stability Analysis of ODE

• **Lyapunov stability:**
  Consider system \( \dot{x}(t) = f(x(t)), \ x(0) = x_0 \).
  \( V(x) \) be scalar function having continuous first derivatives, satisfying
  1. \( V(x) \) is positive definite
  2. \( \frac{dV(x)}{dx} \dot{x} \) is negative definite

  Then the system is **asymptotically stable**

• \( V(x) \) is called **Lyapunov function**

• \( V(x) \) can be a measure for the total **energy** in the system

• Decreasing “energy derivative” and lower bound on energy imply (asymptotic) stability
Lyapunov Stability Analysis of Nonlinear ODE

- **Example:** regular pendulum (mass $m$, length $l$, autonomous) with damping (parameter $k$). Dynamics can be derived from example of inverted pendulum on cart (see Lec. 1), with $\theta \rightarrow \theta + \pi$, no cart, no input:

  $$ml^2\ddot{\theta} + k\dot{\theta} + mgl\sin\theta = 0$$

  pick $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

- Select $V(x)$ as the sum of potential and kinetic energy:

  $$V(x) = mgl(1 - \cos \theta) + \frac{1}{2}ml^2\dot{\theta}^2$$
• Compute time derivative of $V(x)$ along trajectory:

$$\frac{dV}{dx} \dot{x} = \dot{\theta}(mgl \sin \theta + ml^2 \ddot{\theta}) = -k\dot{\theta}^2$$

• Observe that:

- $V(x) > 0, \ V(0) = 0$
- $\frac{dV}{dx} \dot{x} < 0 \implies$ the system is asymptotically stable

• Consider now the case with no damping, $k = 0 \implies$ the system is stable
\begin{equation*}
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.1x_1 - x_2
\end{aligned}
\end{equation*}

\begin{equation*}
V(x) = x^T P x, \quad P = \begin{bmatrix}
10.5 & -0.5 \\
-0.5 & 0.5
\end{bmatrix}
\end{equation*}
Lyapunov Stability Analysis of Linear ODE

- Consider the system \( \dot{x}(t) = Ax(t), \ x(0) = x_0 \)

- A useful choice is the Lyapunov function \( V(x) := x^T P x \) (introduced before), where \( P = P^T \in \mathbb{R}^{n \times n} \) is positive definite. Compute:

\[
\frac{dV}{dx} \dot{x} = \dot{x}^T P x + x^T P \dot{x} = (Ax)^T P x + x^T P A x
= x^T A^T P x + x^T P A x = x^T (A^T P + PA) x
= -x^T Q x
\]

- Here \( Q \) is defined as \( Q := -(A^T P + PA) \)
  As a result: \( Q > 0 \Rightarrow \) model is asymptotically stable
Lyapunov Stability Analysis of Linear ODE

- **Lyapunov stability**, verification technique:
  1. Select $Q = Q^T > 0$
  2. Let $P$ be a symmetric solution to the matrix equation

\[ A^T P + PA = -Q \]

(a.k.a. Lyapunov equation)
  3. Then the model is asymptotically stable if and only if the resulting $P$ satisfies $P > 0$

- Result holds for any $Q$ with $Q = Q^T > 0$

- in MATLAB, command `lyap`